

# Solutions of the reflection equation for face and vertex models associated with $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ and $A_n^{(2)}$

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## Abstract

We present new diagonal solutions of the reflection equation for elliptic solutions of the star-triangle relation. The models considered are related to the affine Lie algebras  $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$  and  $A_n^{(2)}$ . We recover all known diagonal solutions associated with these algebras and find how these solutions are related in the elliptic regime. Furthermore, new solutions of the reflection equation follow for the associated vertex models in the trigonometric limit.

## 1 Introduction

Much work has recently been done in integrable quantum field theory [1, 2, 3, 4, 5] and lattice statistical mechanics [6, 7, 8, 9] on models with a boundary, where integrability manifests itself via solutions of the reflection equation (RE) [1, 6]. In field theory, attention is focused on the boundary  $S$ -matrix. In statistical mechanics, the emphasis has been on deriving solutions of the RE and the calculation of various surface critical phenomena, both at and away from criticality (see, e.g. [8, 9] for recent reviews).

Integrable models exhibit a natural connection with affine Lie algebras [10, 11, 12, 13, 14]. Our interest here lies in both the vertex and face formulation of models associated with the algebras  $X_n^{(1)} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$  and  $A_n^{(2)}$ . These higher rank models include some well known models as special cases. In particular, the Andrews-Baxter-Forrester model (ABF) [15] is related to  $A_1^{(1)}$  while the dilute  $A_L$  models [16], which contain an Ising model in a magnetic field for  $A_3$  [16], are related to  $A_2^{(2)}$ . The excess surface critical exponents  $\alpha_s$  and  $\delta_s$  were recently derived for the dilute  $A_L$  models, including the Ising values  $\alpha_s = 1$  at  $L = 2$  and  $\delta_s = -\frac{15}{7}$  at  $L = 3$  [17].

Here we consider the RE for the higher rank models with a view to deriving new surface critical phenomena. The RE was first written down in field theoretic language [1], later for vertex models [6] and more recently in the interaction-round-a-face (IRF) formulation [18, 19, 20, 7, 21]. We have solved the RE for the above elliptic face models, and in so doing have also

obtained new solutions in the vertex limit. We find an explicit connection between solutions found in various limits which were thought isolated before.

## 2 IRF formulation and vertex correspondence

We begin by fixing our notation and recalling the basic ingredients for integrability, both in the bulk and at a boundary. We will refer to solutions of the star-triangle equation (STR) as bulk weights. Given the bulk weights, we refer to the associated solutions of the RE as boundary weights. Our representation of the bulk and boundary weights is

$$\mu \begin{array}{c} \kappa \\ \boxed{u} \\ \nu \end{array} \sigma = W \left( \begin{array}{cc} a & a + \hat{\kappa} \\ a + \hat{\mu} & a + \hat{\mu} + \hat{\nu} \end{array} \middle| u \right), \quad (\hat{\mu} + \hat{\nu} = \hat{\kappa} + \hat{\sigma}) \quad (1)$$

$$\begin{array}{c} \nu \\ \triangleleft u \\ \mu \end{array} = K \left( \begin{array}{cc} a + \hat{\nu} & \\ a & a + \hat{\mu} \end{array} \middle| u \right). \quad (2)$$

We use the same algebraic notation as in [13, 14], with Latin letters  $(a, b, \dots)$  for states, Greek letters  $(\mu, \nu, \dots)$  for elementary vectors and  $u$  as spectral parameter. The corner triangle locates the state associated with the square (bulk) and triangular (boundary) face (assumed to be  $a$  unless otherwise stated). In the arrow representation [20] it is the point from which the arrows emanate. This is made clear in the following relations, which also show the vertex-face correspondence in the critical (trigonometric) limit,

$$\begin{array}{c} \delta \\ \alpha \boxed{u} \gamma \\ \beta \end{array} = \alpha \begin{array}{c} \delta \\ \boxed{u} \\ \beta \end{array} \gamma \xrightarrow{“|a| \rightarrow \infty”} \alpha \begin{array}{c} \delta \\ \text{---} \text{---} \text{---} \\ u \\ \text{---} \text{---} \text{---} \\ \beta \end{array} \gamma = R_{\alpha\beta}^{\delta\gamma}(u), \quad (3)$$

$$\begin{array}{c} \beta \\ \triangleleft u \\ \alpha \end{array} = \begin{array}{c} \beta \\ \triangleleft u \\ \alpha \end{array} \xrightarrow{“|a| \rightarrow \infty”} \begin{array}{c} \beta \\ \text{---} \text{---} \text{---} \\ u \\ \text{---} \text{---} \text{---} \\ \alpha \end{array} = K_{\alpha}^{\beta}(u).$$

Here the limit “ $|a| \rightarrow \infty$ ” will be specified more concretely in Sec. 5. The STR has the graphical form

$$\begin{array}{c} \text{Hexagon 1} \end{array} = \begin{array}{c} \text{Hexagon 2} \end{array} \quad (4)$$

(The hexagons are drawn with vertices at the corners of a regular hexagon. Hexagon 1 has internal edges labeled  $u$  (top-left),  $v$  (bottom-left), and  $u-v$  (middle). Hexagon 2 has internal edges labeled  $u-v$  (top-left),  $u$  (bottom-left), and  $v$  (middle). A full dot is placed at the center of each hexagon.)

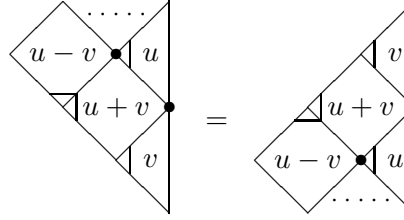
where the edges (sites) of the outer hexagon take on the same elementary vectors (states) on either side of the relation and the internal edges (sites) are summed over (represented by a full dot). Once a configuration of vector differences is specified on the outer hexagon, only one state ( $f$  say at the top left corner) is required to specify the others (as with any configuration of meeting faces).

If we assign states anticlockwise in alphabetical order from the left most corner of the above hexagon, with summed height  $g$  in the centre and state  $a$  at the start, then the STR reads

$$\sum_g W \left( \begin{array}{cc|c} f & e & u \\ a & g & \end{array} \right) W \left( \begin{array}{cc|c} a & g & v \\ b & c & \end{array} \right) W \left( \begin{array}{cc|c} e & d & u-v \\ g & c & \end{array} \right) =$$

$$\sum_g W \left( \begin{array}{cc|c} f & g & u-v \\ a & b & \end{array} \right) W \left( \begin{array}{cc|c} g & d & u \\ b & c & \end{array} \right) W \left( \begin{array}{cc|c} f & e & v \\ g & d & \end{array} \right). \quad (5)$$

The RE has the graphical form



$$= \quad (6)$$

where the external edges of same state on each side of the relation carry the same vector differences and internal edges are summed. The edges connected by dashes are identified as one internal edge.

If we assign states as before but starting from the top most corner and sum in both  $f$  and  $g$  we have the RE in the form

$$\sum_{fg} W \left( \begin{array}{cc|c} c & b & u-v \\ g & a & \end{array} \right) K \left( \begin{array}{cc|c} a & & u \\ g & f & \end{array} \right) W \left( \begin{array}{cc|c} c & g & u+v \\ d & f & \end{array} \right) K \left( \begin{array}{cc|c} f & & v \\ d & e & \end{array} \right) =$$

$$\sum_{fg} K \left( \begin{array}{cc|c} a & & v \\ b & f & \end{array} \right) W \left( \begin{array}{cc|c} c & b & u+v \\ g & f & \end{array} \right) K \left( \begin{array}{cc|c} g & f & u \\ & e & \end{array} \right) W \left( \begin{array}{cc|c} c & g & u-v \\ d & e & \end{array} \right). \quad (7)$$

This formulation follows that given in [19, 20, 7, 21]. It is equivalent to the formulation of [18] in the special case that a crossing symmetry of the bulk weights exists. Such crossing symmetry is absent in the higher rank  $A_{n>1}^{(1)}$  models [12] and as a result the formulation given in [18] does not support diagonal boundary weight solutions for these models.

The above RE (7) reduces to the original formulation [1, 6]

$$R_{12}(u-v)K_1(u)R_{21}(u+v)K_2(v) = K_2(v)R_{21}(u+v)K_1(u)R_{12}(u-v), \quad (8)$$

in the vertex limit (3).

### 3 Bulk face weights

Consider the elliptic face models associated with  $X_n^{(1)}$  [13] and  $A_n^{(2)}$  [14]. In these models, the states range over the dual space of the Cartan subalgebra of  $X_n^{(1)}$  and  $\hat{X}_n^{(1)}$  [14], respectively. Arrows  $\alpha, \beta, \mu, \nu$ , etc run over the set

$$\begin{aligned} \{1, 2, \dots, n+1\} & \quad \text{for } A_n^{(1)}, \\ \{1, 2, \dots, n, 0, -n, \dots, -1\} & \quad \text{for } B_n^{(1)}, A_{2n}^{(2)}, \\ \{1, 2, \dots, n, -n, \dots, -1\} & \quad \text{for } C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}. \end{aligned} \quad (9)$$

In particular,  $\sum_\kappa, \prod_\beta$ , etc are to be taken over the above set. In terms of the notation of [13, 14] the bulk weights are given by

$$\mu \begin{array}{|c|} \hline \mu \\ \hline \end{array} \begin{array}{|c|} \hline u \\ \hline \end{array} \mu = \frac{[1+u]}{[1]}, \quad \mu \begin{array}{|c|} \hline \mu \\ \hline \end{array} \begin{array}{|c|} \hline u \\ \hline \end{array} \nu = \frac{[a_\mu - a_\nu - u]}{[a_\mu - a_\nu]} \quad (\mu \neq \nu),$$

$$\mu \begin{array}{|c|} \hline \nu \\ \hline \end{array} \begin{array}{|c|} \hline u \\ \hline \end{array} \mu = \frac{[u]}{[1]} \left( \frac{[a_\mu - a_\nu + 1][a_\mu - a_\nu - 1]}{[a_\mu - a_\nu]^2} \right)^{1/2} \quad (\mu \neq \nu).$$

for  $A_n^{(1)}$ , while for  $B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_n^{(2)}$  they read

$$\mu \begin{array}{|c|} \hline \mu \\ \hline \end{array} \begin{array}{|c|} \hline u \\ \hline \end{array} \mu = \frac{[\lambda - u][1 + u]}{[\lambda][1]} \quad (\mu \neq 0),$$

$$\mu \begin{array}{|c|} \hline \mu \\ \hline \end{array} \begin{array}{|c|} \hline u \\ \hline \end{array} \nu = \frac{[\lambda - u][a_\mu - a_\nu - u]}{[\lambda][a_\mu - a_\nu]} \quad (\mu \neq \pm \nu),$$

$$\mu \begin{array}{|c|} \hline \nu \\ \hline \end{array} \begin{array}{|c|} \hline u \\ \hline \end{array} \mu = \frac{[\lambda - u][u]}{[\lambda][1]} \left( \frac{[a_\mu - a_\nu + 1][a_\mu - a_\nu - 1]}{[a_\mu - a_\nu]^2} \right)^{1/2} \quad (\mu \neq \pm \nu),$$

$$\mu \begin{array}{|c|} \hline \nu \\ \hline \end{array} \begin{array}{|c|} \hline u \\ \hline \end{array} - \nu = \frac{[u][a_\mu + a_\nu + 1 + \lambda - u]}{[\lambda][a_\mu + a_\nu + 1]} (G_{a,\mu} G_{a,\nu})^{1/2} \quad (\mu \neq \nu),$$

$$\begin{aligned} \mu \begin{array}{|c|} \hline \mu \\ \hline \end{array} \begin{array}{|c|} \hline u \\ \hline \end{array} - \mu &= \frac{[\lambda + u][2a_\mu + 1 + 2\lambda - u]}{[\lambda][2a_\mu + 1 + 2\lambda]} - \frac{[u][2a_\mu + 1 + \lambda - u]}{[\lambda][2a_\mu + 1 + 2\lambda]} H_{a,\mu}, \\ &= \frac{[\lambda - u][2a_\mu + 1 - u]}{[\lambda][2a_\mu + 1]} + \frac{[u][2a_\mu + 1 + \lambda - u]}{[\lambda][2a_\mu + 1]} G_{a,\mu} (\mu \neq 0). \end{aligned}$$

In the above we remind the reader that  $a_\mu = -a_{-\mu}$  for all  $\mu$  in (9) except  $a_0 = -1/2$  [13, 14]. The crossing parameter is given by  $\lambda = -tg/2$  for  $X_n^{(1)}$  and  $\lambda = -g/2 + L/2$  for  $A_n^{(2)}$  (note that  $\lambda$  is shifted by  $L/2$  from [14]). The parameters  $t, g$  are given in Table 1. Here  $L$  is arbitrary for the unrestricted solid-on-solid (SOS) models but will be specified later for the restricted (RSOS) models. We have further defined

$$[u] = [u, p] = \vartheta_1(\pi u/L, p), \quad (10)$$

where

$$\vartheta_1(u, p) = 2|p|^{1/8} \sin u \prod_{j=1}^{\infty} (1 - 2p^j \cos 2u + p^{2j})(1 - p^j) \quad (11)$$

is a standard elliptic theta-function of nome  $p = e^{2\pi i \tau}$ . For convenience we also use

$$[u, p]' = \vartheta_4(\pi u/L, p), \quad (12)$$

where

$$\vartheta_4(u, p) = \prod_{j=1}^{\infty} (1 - 2p^{j-1/2} \cos 2u + p^{2j-1})(1 - p^j). \quad (13)$$

The quantity  $G_{a,\mu}$  is determined by  $G_{a,\mu} = G_{a+\hat{\mu}}/G_a$  ( $\mu \neq 0$ ) and  $G_{a,0} = 1$ , where

$$\begin{aligned} G_a &= \prod_{1 \leq i < j \leq n+1} [a_i - a_j] \quad \text{for } A_n^{(1)}, \\ &= \varepsilon(a) \prod_{i=1}^n h(a_i) \prod_{1 \leq i < j \leq n} [a_i - a_j][a_i + a_j] \quad \text{otherwise.} \end{aligned} \quad (14)$$

The sign factor  $\varepsilon(a)$  is such that  $\varepsilon(a + \hat{\mu})/\varepsilon(a) = -1$  for  $C_n^{(1)}$  and  $A_{2n-1}^{(2)}$  only and is unity for the other cases. The function  $h(a)$  is given in Table 1. Finally,

$$H_{a,\mu} = \sum_{\kappa \neq \mu} \frac{[a_\mu + a_\kappa + 1 + 2\lambda]}{[a_\mu + a_\kappa + 1]} G_{a,\kappa}. \quad (15)$$

**Table 1**

type	$A_n^{(1)}$	$B_n^{(1)}$	$C_n^{(1)}$	$D_n^{(1)}$	$A_{2n}^{(2)}$	$A_{2n-1}^{(2)}$
$g$	$n+1$	$2n-1$	$n+1$	$2n-2$	$2n+1$	$2n$
$t$	1	1	2	1	1	2
$h(a)$	1	$[a]$	$[2a]$	1	$[a][2a, p^2]'$	$[2a, p^2]$

For the face models two inversion relations are satisfied by the bulk weights [13, 14],

$$\sum_g W \left( \begin{matrix} a & g \\ b & c \end{matrix} \middle| u \right) W \left( \begin{matrix} a & d \\ g & c \end{matrix} \middle| -u \right) = \delta_{bd} \varrho_1(u), \quad (16)$$

$$\sum_g \left( \frac{G_g G_b}{G_a G_c} \right) W \left( \begin{matrix} g & c \\ a & b \end{matrix} \middle| \lambda - u \right) W \left( \begin{matrix} g & a \\ c & d \end{matrix} \middle| \lambda + u \right) = \delta_{bd} \varrho_2(u). \quad (17)$$

Here the inversion functions are given by

$$\varrho_1(u) = \frac{[1+u][1-u]}{[1]^2}, \quad \varrho_2(u) = \frac{[\lambda+u][\lambda-u]}{[1]^2} \quad \text{for } A_n^{(1)}, \quad (18)$$

$$\varrho_1(u) = \varrho_2(u) = \frac{[\lambda+u][\lambda-u][1+u][1-u]}{[\lambda]^2[1]^2} \quad \text{otherwise.} \quad (19)$$

## 4 Boundary face weights

Given the above solutions of the STR (5), we have found that the RE (7) has the diagonal face solutions

$$K \left( a + \hat{\mu} \quad a \middle| u \right) = \frac{[a_\mu - \eta + u]}{[a_\mu - \eta - u]} f_a(u) \delta_{ab}, \quad (20)$$

For  $A_n^{(1)}$ ,  $\eta$  is a free parameter. However, for  $B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$  and  $A_n^{(2)}$ , it must be chosen as

$$\eta = -\frac{\lambda + 1}{2} + \frac{rL}{2} + \frac{sL\tau}{2}, \quad r, s \in \mathbf{Z}. \quad (21)$$

The two solutions (20) corresponding to  $(r, s)$  and  $(r', s')$  are in fact different only by an overall function of  $u$  if  $r - r' \equiv s - s' \equiv 0 \pmod{2}$ . The function  $f_a(u)$  is not restricted from the RE (7). However, it may be normalized as

$$f_a(u) = z_a(u) e^{-2\pi i \omega s u / L} h(u + \eta) \prod_{\beta} [a_\beta - \eta - u], \quad (22)$$

where  $\omega$  is defined by

$$\omega = \begin{cases} 0 & A_1^{(1)} \\ \lambda & C_n^{(1)}, D_n^{(1)} \\ \lambda - 1/2 & B_n^{(1)} \\ \lambda - 1/2 - L/2 & A_{2n}^{(2)} \\ \lambda - L/2 & A_{2n-1}^{(2)} \end{cases} \quad (23)$$

and  $z_a(u)$  is any function satisfying  $z_a(u + \lambda) = z_a(-u)$ . Then the solutions (20) further fulfill the boundary crossing relation (except  $A_{n>1}^{(1)}$ )

$$\sum_g \left( \frac{G_g}{G_b} \right)^{1/2} W \left( \begin{matrix} a & g \\ b & c \end{matrix} \middle| 2u + \lambda \right) K \left( \begin{matrix} a & \\ g & c \end{matrix} \middle| u + \lambda \right) = \varrho_3(u) K \left( \begin{matrix} a & \\ b & c \end{matrix} \middle| -u \right). \quad (24)$$

Here the boundary crossing function  $\varrho_3(u)$  is given by

$$\varrho_3(u) = \begin{cases} \frac{[2 - 2u]}{[1]} & \text{for } A_1^{(1)}, \\ \frac{[2u + 2\lambda][1 - \lambda - 2u]}{[1][\lambda]} & \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)} \text{ and } A_n^{(2)}. \end{cases} \quad (25)$$

The proof of (7) and (24) for (20) is similar to that for the STR in [13]. In the sequel we shall exclusively consider the cases  $r, s = 0, 1$  without loss of generality.

## 5 From face to vertex weights

In the limit  $p \rightarrow 0$ ,  $|a_\mu| \rightarrow \infty$  ( $\mu \neq 0$ ) the bulk weights in Sec. 3 reduce, up to normalization, to the vertex Boltzmann weights given in appendix A of [14] for  $A_{2n-1}^{(2)}$  and in [10, 11] for the other

algebras. In the correspondence (3), the indices (9) should be identified, including their orders, with  $1, 2, \dots, N$  in [10]. The parameters  $k = e^{-2h}$ ,  $x, \xi$  in [10] and  $L, u, \lambda$  here are related as  $h = i\pi/2L$ ,  $k = e^{-i\pi/L}$ ,  $x = e^{2\pi i u/L}$ ,  $\xi = e^{2\pi i \lambda/L}$ . In terms of these variables, the limit is to be taken so that  $|k^{a_i}| \ll |k^{a_j}|$ , ( $i < j$ ) for  $A_n^{(1)}$  and  $|k^{a_1}| \ll |k^{a_2}| \ll \dots \ll |k^{a_n}| \ll 1$  for the other algebras. Proceeding in the same manner for the boundary weights, we obtain new diagonal  $K$ -matrices that satisfy the RE (8) and the trigonometric limits of (24) and (25). Below we shall present them using the function  $\epsilon_z = -1 + 2|\text{sgn}(z)|$  with  $\text{sgn}(z) = 1, 0$  and  $-1$  for  $z > 0, z = 0$  and  $z < 0$ , respectively.

The RE for  $A_n^{(1)}$  has the diagonal vertex solutions

$$K_\alpha^\alpha(u) = F(u)e^{4\text{sgn}(\alpha-\kappa)hu} \sinh[2h(\phi + \epsilon_{\alpha-\kappa}u)], \quad (26)$$

where  $\phi \in \mathbf{C}$ ,  $\kappa \in \mathbf{R}$  and  $F(u)$  are arbitrary. Note that for non-integer  $\kappa$  there are essentially only exponential solutions. Since any RE in the vertex limit involves only two boundary weights and the sign of their state difference, the following (cf.  $\kappa = 1$  or  $n+1$ ) solution is also admissible

$$K_\alpha^\alpha(u) = \begin{cases} F(u)e^{-2hu} \sinh[2h(\phi + u)] & \alpha \leq \kappa, \\ F(u)e^{2hu} \sinh[2h(\phi - u)] & \alpha > \kappa. \end{cases} \quad (27)$$

This recovers the solution for the  $A_n^{(1)}$  vertex model given in [23] if we take  $h = 1/2$ . The function  $F(u)$  is not restricted by the RE. However for  $A_1^{(1)}$  ( $\lambda = -1$ ), its appropriate choice renders the vertex analogue of (24) still valid with the same  $\varrho_3(u)$  and  $[u] \propto \sinh(2hu)$ . Up to an overall factor  $z(u)$  obeying  $z(u-1) = z(-u)$ , there are three such normalized solutions. One of them is given by putting  $\kappa = 1$  and  $F(u) = 1$  in (27). The other two are  $(K_1^1(u), K_2^2(u)) = (1, 1)$  and  $(e^{-4hu}, e^{4hu})$ .

For the remaining algebras we classify the vertex limit according to whether the integer  $s$  in (21) is 0 or 1.

$s = 0$ .

$$K_\alpha^\alpha(u) = \begin{cases} F(u)e^{-4\text{sgn}(\alpha)hu} \sinh[2h(\frac{\lambda}{2} - \epsilon_\alpha u) + \frac{i\pi r}{2}] \sinh[2h(\frac{\lambda+1}{2} - u) + \frac{i\pi r}{2}] & B_n^{(1)}, A_{2n}^{(2)} \\ F(u)e^{-4\text{sgn}(\alpha)hu} \sinh[4h(\frac{\lambda+1}{2} - u)] & C_n^{(1)}, A_{2n-1}^{(2)} \\ F(u)e^{-4\text{sgn}(\alpha)hu} & D_n^{(1)} \end{cases} \quad (28)$$

$s = 1$ . In this case we have only  $K$  matrices which are multiples of the identity:

$$K_\alpha^\alpha(u) = \begin{cases} F(u) & B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)} \\ F(u) \sinh[4h(\frac{\lambda+1}{2} - u)] & C_n^{(1)}, A_{2n}^{(2)} \end{cases} \quad (29)$$

As in (26) and (27),  $F(u)$  is not restricted from the RE in (28) and (29). However, the simple choice  $F(u) = 1$  (up to a factor  $z(u)$  satisfying  $z(u+\lambda) = z(-u)$ ) allows the vertex analogue of (24) to hold with again the same  $\varrho_3(u)$  and  $[u] \propto \sinh(2hu)$ .

All the solutions in [24] can be recovered from our solution for  $A_2^{(2)}$ . Besides the identity solution ( $s = 1$ ), (28) in this case is proportional to

$$K_{-1}^{-1}(u) = e^{4hu}, \quad K_0^0(u) = \frac{\sinh[2h(\frac{\lambda}{2} + u) + \frac{\pi i r}{2}]}{\sinh[2h(\frac{\lambda}{2} - u) + \frac{\pi i r}{2}]} \quad \text{and} \quad K_1^1(u) = e^{-4hu}. \quad (30)$$

This can be identified with the two nontrivial solutions in [24].

## 6 Restricted models

The RSOS models follow in a natural way from the unrestricted models that we have discussed so far. To do this, one sets  $L = t(l + g)$  for  $X_n^{(1)}$  ( $L = t(l + \dot{g})$  for  $X_n^{(2)}$ ) where  $t, g$  are given in Table 1 and  $l$  is a positive integer. The local state  $a$  is taken as a level  $l$  dominant integral weight of  $X_n^{(1)}$  ( $X_n^{(1)}$  for  $X_n^{(2)}$ ). One also imposes special adjacency rules on the states (see [13, 14] for details). Then it has been shown that the restricted bulk weights are finite and satisfy the STR (5) and the inversion relations among themselves. We have proved that all such features are also valid in the boundary case. Namely, under the restriction the boundary face weights (20), (22) (with the simple choice  $z_a(u) = 1$  for example) are finite and satisfies the RE (7) and boundary crossing relation (24) among themselves. The only previous boundary solutions for such RSOS models were for the ABF model [18, 21] and for the dilute  $A_L$  models built on  $A_2^{(2)}$  [17].

## 7 Summary and conclusion

We have derived elliptic diagonal solutions of the RE for the face models related to the affine Lie algebras  $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$  and  $A_n^{(2)}$ . For  $A_n^{(1)}$  we found a free parameter  $\eta$  which also survives in the vertex limit in agreement with previous work [23]. For the other algebras we found that  $\eta$  takes only discrete values. This led to four inequivalent forms of solution corresponding to  $r, s \in \mathbf{Z}_2$ . Of these four, the two solutions for  $(r, s) = (0, 1)$  and  $(1, 1)$  degenerated to the identity upon taking the vertex limit.<sup>1</sup> This explains the presence of only three distinct solutions in previous work [24] and shows their connection lies in the elliptic regime. An analogous set of three solutions exist for  $G_2^{(1)}$  [25]. For the face models, previously known solutions were for  $A_1^{(1)}$  [18, 21] (see also [22] for related work) and  $A_2^{(2)}$  [17]. These are easily recovered from those given here. The transfer matrix eigenspectra for the diagonal  $K$ -matrices can be obtained in the vertex limit by means of the Bethe Ansatz, as has been done already for a number of models (see, e.g. [6, 27, 28, 29, 25, 30, 31] and refs therein). It is known that different  $K$ -matrices lead to different universality classes of surface critical behaviour [32, 30]. The investigation of the higher rank solutions obtained here is thus of considerable interest. The elliptic solutions for the higher rank face models are also of interest from the point of view of calculating physical quantities, such as surface critical exponents, away from criticality.

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<sup>1</sup>The solution  $K = 1$  had previously been known for the higher rank cases and leads to quantum group invariant spin chains [24, 26].



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